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Chaos resulting from the application of a weak, slowly oscillating electric field to a wedge of smectic C liquid crystal

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Abstract. This article contains the theoretical study of a weak, slowly oscillating electric field across a wedge of non-chiral smectic C liquid crystal. Smectic C continuum theory is used to derive the governing equations. By use of the method due to Melnikov these equations are then examined for the possibility of exhibiting chaotic solutions. The results are discussed in relation to possible experimental observations and material constants.

1. Introduction

Liquid crystals consist of elongated molecules which have a local preferred direction. Smectic C liquid crystals are layered structures in which the average long molecular axis lies tilted at an angle θ to the layer normal. This tilt angle is dependent upon the temperature of the system which, for our purposes, is assumed constant. Apart from possessing geometric anisotropy the constituent molecules are also dielectrically anisotropic. Under the application of an external electric field this may cause the molecules to rotate within the layers; applications of strong electric fields may lead to layer distortions.

It is common in liquid crystal theory to describe the average direction of the long molecular axis within a sample by a unit vector $\boldsymbol{n}$, usually called the director. Following the description by de Gennes [1] we employ two unit vectors to describe a smectic C. The orientation of the parallel smectic layers is described by the unit layer normal $\boldsymbol{a}$. The second unit vector $\boldsymbol{c}$ which is perpendicular to $\boldsymbol{a}$ (and therefore tangential to the layers) is the unit orthogonal projection of the n-director onto the smectic planes. Mathematically it is convenient to introduce a third vector $\boldsymbol{b}$ as

$$\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c}\,. \tag{1}$$

These vectors are often called the a-, c- and b-directors. To describe the c- and b-directors it is convenient to introduce ϕ as the angle that the c-director makes with some fixed direction within the plane of the layers.

In the absence of any defects, the unit layer normal fulfils the following constraint, due to Oseen [2],

$$\nabla \times \boldsymbol{a} = 0\,. \tag{2}$$

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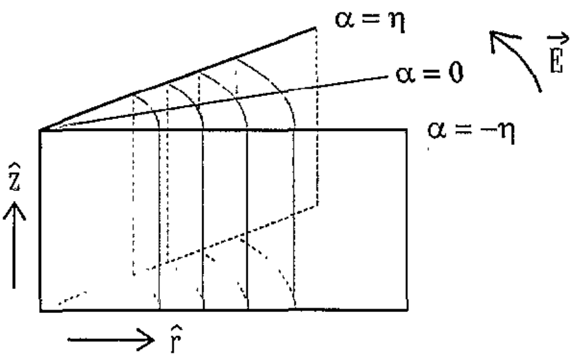

Figure 1. The smectic planes as parts of concentric cylinders with their common axes coinciding with the apex of the wedge, shown with the cylindrical polar coordinate system. The electric field E is applied across the bounding plates at $\alpha = \pm\eta$ and follows the curve of the smectic planes.

This has been known to restrict the possible layer configurations to planes, cylinders, spheres, circular tori, Dupin cyclides and parabolic cyclides [3].

It is assumed throughout this work that the smectic layers remain intact as parts of concentric cylinders and that there is no bulk flow in the sample. Consequently the only motion occurs in rotations of the molecules within the layered structure. In a cylindrical coordinate system $\{r, \alpha, z\}$ we employ the geometry used by Carlsson *et al* [4] and form a wedge of liquid crystal by placing boundaries at $\alpha = \pm\eta$, applying the voltage across these. The apex of the wedge is such that it coincides with the centres of the concentric cylinders and the z-axis of the cylindrical polar coordinate system, as depicted in figure 1. Thus we may set

$$\begin{aligned} \boldsymbol{a} &= (1, 0, 0) \\ \boldsymbol{c} &= (0, \sin\phi, \cos\phi) \\ \boldsymbol{b} &= (0, -\cos\phi, \sin\phi) \end{aligned} \tag{3}$$

as our ansatz for the three directors.

We consider a weak, slowly oscillating potential difference across the plates:

$$V(t) = U\sqrt{\omega}\cos(\omega t) \tag{4}$$

where the angular frequency ω is small and the maximum potential difference is

$$V_0 = U\sqrt{\omega}\,. \tag{5}$$

This gives rise to an electric field of the form

$$\boldsymbol{E}(t) = \frac{U}{2r\eta}\sqrt{\omega}\cos(\omega t)\widehat{\alpha}\,. \tag{6}$$

This particular form for an oscillatory field has been chosen for reasons which will be discussed in section 4.

In section 2 we employ the smectic C continuum theory of Leslie *et al* [3] to derive the differential equation governing the motion of twist walls in the sample. In section 3 we introduce and use Melnikov's method to examine this equation for chaotic behaviour. This approach has recently been successfully used by Stewart *et al* [5] in the study of a static field attenuated by a weak, slowly oscillating field applied at a small angle to the layers of a planar-aligned sample of smectic C liquid crystal. As discussed in [5], a twist wall (or travelling wave) will propagate through the system, driven by elastic and dielectric torques. The original twist is formed by competing boundary condititions. The twist will prefer to travel such that the c-director aligns with the direction which minimizes the energy of the system. For a full discussion of the effects of competing torques on travelling wave solutions the reader is referred to [5].

The aim of this article is to examine the possibility of the application of a field of the given type resulting in chaotic motion of the twist walls within the sample. Since our interest lies in the existence of chaos at a local level it is sufficient to examine small sections of the sample separately. By exploiting this fact in section 2 the equations are greatly simplified.

2. Continuum equations

In terms of $\boldsymbol{a}$ and $\boldsymbol{c}$, the relevant continuum equations are given by Leslie *et al* [3] to be

$$\boldsymbol{\Pi}^a + \tilde{\boldsymbol{g}}^a + \lambda \boldsymbol{a} + \mu \boldsymbol{c} + \nabla \times \boldsymbol{\beta} = \boldsymbol{0} \tag{7}$$

$$\boldsymbol{\Pi}^c + \tilde{\boldsymbol{g}}^c + \mu \boldsymbol{a} + \chi \boldsymbol{c} = \boldsymbol{0} \tag{8}$$

this coupled system representing the conservation of angular momentum. We shall refer to these equations as the a- and c-equations respectively. The terms $\boldsymbol{\Pi}^a$ and $\boldsymbol{\Pi}^c$ are bulk elastic and dieletric terms. The dynamic terms $\tilde{\boldsymbol{g}}^a$ and $\tilde{\boldsymbol{g}}^c$ are related to the intrinsic viscous torque of the system as discussed by Leslie *et al* [3]. It will be seen later that $\boldsymbol{\Pi}^a$ and $\tilde{\boldsymbol{g}}^a$ need not be calculated explicitly. The forms of $\boldsymbol{\Pi}^c$ and $\tilde{\boldsymbol{g}}^c$ are given in the appendix. The Lagrange multipliers $\boldsymbol{\beta}$, λ, χ and μ arise from the constraints (2) and

$$\boldsymbol{a} \cdot \boldsymbol{a} = 1 \tag{9}$$

$$\boldsymbol{c} \cdot \boldsymbol{c} = 1 \tag{10}$$

$$\boldsymbol{a} \cdot \boldsymbol{c} = 0 \tag{11}$$

respectively.

Taking the scalar product of the c-equations (8) with $\boldsymbol{a}$ gives

$$\mu = -\Pi_1^c - \tilde{g}_1^c \,. \tag{12}$$

This makes the first component of the c-equations trivially satisfied. The Lagrange multiplier χ can then be eliminated between the second and third c-equations, leaving just

$$(\Pi_2^c + \tilde{g}_2^c)\cos\phi - (\Pi_3^c + \tilde{g}_3^c)\sin\phi = 0 \,. \tag{13}$$

In common with earlier work on smectics (for example [6]) we can use equation (12) and take the divergence of the resulting a-equations (7) to gain a PDE which λ must solve, namely

$$\frac{\partial}{\partial r}\left(r(\Pi_1^a + \lambda + \tilde{g}_1^a)\right) + \frac{\partial}{\partial \alpha}(\Pi_2^a + \mu\sin\phi + \tilde{g}_2^a) + \frac{\partial}{\partial z}\left(r(\Pi_3^a + \mu\cos\phi + \tilde{g}_3^a)\right) = 0 \,. \tag{14}$$

This obviously has solutions for λ in terms of ϕ by integration with respect to r. The existence of the vector Lagrange multiplier $\boldsymbol{\beta}$, such that the a-equations can be solved completely is then guaranteed by the following identity for any twice-differentiable vector $\boldsymbol{F}$:

$$\nabla \cdot \boldsymbol{F} = 0 \qquad \text{if and only if } \boldsymbol{F} = \nabla \times \boldsymbol{G} \tag{15}$$

for some unique (to within the gradient of an arbitrary scalar field) vector $\boldsymbol{G}$. By writing $\boldsymbol{F} = \boldsymbol{\Pi}^a + \lambda \boldsymbol{a} + \mu \boldsymbol{c}$, the value of λ derived at equation (14) ensures that $\nabla \cdot \boldsymbol{F} = 0$ and so, by setting $\boldsymbol{\beta} = -\boldsymbol{G}$ in the above equation, the a-equations are solved completely for any given ϕ. Thus solving equation (13) completely solves the coupled system (7) and (8).

Noting that the geometry has translational invariance in z we make the assumption that ϕ is independent of z. Furthermore, since we are seeking the existence of chaos within but not necessarily throughout the sample, we may take a local approach. We do this by considering a thin, curved strip of liquid crystal such that r may be considered constant. Thus we have

$$\phi = \phi(\alpha, t) . \tag{16}$$

We now proceed to calculate equation (13) from the expressions (A.5), (A.7) and (A.8) in the appendix. Doing so yields

$$\frac{B}{2}\frac{\partial^2 u}{\partial \alpha^2} + A \sin u \cos u - \lambda_5 r^2 \frac{\partial u}{\partial t} + \tfrac{1}{2}\epsilon_a \epsilon_0 \sin^2\theta \sin u \left(\frac{U\sqrt{\omega}\cos(\omega t)}{2\eta}\right)^2 = 0 \tag{17}$$

where we have set

$$u = 2\phi \tag{18}$$

and A is the combination of elastic constants given by (A.3).

Motivated by previous analyses of twist wall solutions (see, for example, [5] or [7]) we seek solutions of the form

$$u = u(\tau) \tag{19}$$

where

$$\tau = \sqrt{\frac{2}{B}}(\alpha - \alpha_0) + \frac{\omega t}{\lambda_5 r^2} \tag{20}$$

and α_0 is a fixed value of α. If we further restrict our attention to a neighbourhood of α_0 such that

$$\sqrt{\frac{2}{B}}(\alpha - \alpha_0)\lambda_5 r^2 = \mathrm{O}(\omega) \tag{21}$$

then by writing

$$v = \frac{\mathrm{d}u}{\mathrm{d}\tau} \tag{22}$$

and

$$\sigma = \tfrac{1}{2}\epsilon_a \epsilon_0 \sin^2\theta \left(\frac{U}{2\eta}\right)^2 \tag{23}$$

equation (17) becomes, in abstract form,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -A \sin u \cos u \end{pmatrix} + \omega \begin{pmatrix} 0 \\ v - \sigma \sin u \cos^2(\lambda_5 r^2 \tau) \end{pmatrix} + \mathrm{O}(\omega^2) \tag{24}$$

by use of equations (17) to (21) and the formula for $\cos(a+b)$. Notably the simplification (20) complicates the inclusion of boundary conditions at $\alpha = \pm\eta$. We overcome this by limiting ourselves to areas of the sample (described by r, α_0 and equation (21)) where $u^0(\alpha, t) = 0$ holds at some time t. We can then redefine time in each such region such that it is possible to set $u^0(0) = 0$ without loss of generality. In doing this we are able to neglect any boundary conditions. In the presence of certain types of boundary conditions it is likely that there would be regions in the sample in which $u^0(\alpha, t)$ remains non-zero for all t. To include such areas in our analysis would cause unnecessary complication.

3. The existence of chaotic twist wall motion

In this section we shall describe and use the Melnikov method for proving the existence of chaotic solutions. For further details of this method the reader is referred to [8], [9], [10] or [11]. To clarify notation we shall rewrite equation (24) as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \boldsymbol{f}(\boldsymbol{x}) + \omega\,\boldsymbol{g}(\boldsymbol{x}, \tau, \omega) + \mathrm{O}(\omega^2) \tag{25}$$

where $\boldsymbol{x} = (u, v)$ and $\boldsymbol{f}$ and $\boldsymbol{g}$ are defined in an obvious manner. To apply the Melnikov method we first note that the unperturbed system ($\omega = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -A \sin u \cos u \end{pmatrix} \tag{26}$$

is Hamiltonian. The Hamiltonian $H(u, v)$ is constant as the system is conservative and is defined by

$$2H = v^2 - A\cos^2 u \tag{27}$$

since $\frac{\partial H}{\partial v} = v$ and $-\frac{\partial H}{\partial u} = -A\sin u \cos u$. Rearranging the above gives

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 = 2H + A\cos^2 u\,. \tag{28}$$

By equation (A.4) we note that $A > 0$, in which case two different types of behaviour can be seen to exist: when $H < 0$, v is not defined for all u, since v^2 is negative for some u and the c-director is oscillating. For $H > 0$, v is defined for all u, since v^2 is always positive and the c-director is rotating. The seperatrix between these types of behaviour, $H = 0$, describes the homoclinic behaviour. It is with the homoclinic solution that the Melnikov method is concerned. By rewriting (28) with $H = 0$ we gain the following expression for the homoclinic solution:

$$\begin{aligned}\left(v^0(\tau)\right)^2 &= \left(\frac{\mathrm{d}u^0}{\mathrm{d}\tau}\right)^2 \\ &= A\,\cos^2 u^0\,.\end{aligned} \tag{29}$$

With the simplification $u^0(0) = 0$, this may be integrated from 0 to τ to yield the upper and lower seperatrices in the phase plane

$$(u^0_\pm(\tau), v^0_\pm(\tau)) = \Big(\pm\sin^{-1}(\tanh(\sqrt{A}\tau))\,,\ \pm\sqrt{A}\operatorname{sech}(\sqrt{A}\tau)\Big)\,. \tag{30}$$

For both branches of the above solution we define the Melnikov function as

$$M_\pm(\tau_0) = \int_{-\infty}^{\infty} \boldsymbol{f}(u^0_\pm(\tau), v^0_\pm(\tau)) \wedge \boldsymbol{g}(u^0_\pm(\tau), v^0_\pm(\tau), \tau + \tau_0)\,\mathrm{d}\tau \tag{31}$$

where $\boldsymbol{f} \wedge \boldsymbol{g} = f_1 g_2 - f_2 g_1$.

For a system of the form (25) for which the unperturbed equation is Hamiltonian and possesses homoclinic (or heteroclinic) solutions to one (or more) hyperbolic saddle points, Melnikov's theorem may be stated as follows.

Melnikov's Theorem. If $M_+(\tau_0)$ or $M_-(\tau_0)$ has simple zeros and is independent of ω, then for $\omega > 0$ sufficiently small the perturbed system (25) exhibits chaos (in the sense of Smale Horseshoes).

Thus, the existence of chaos is guaranteed by the presence of values $\bar{\tau}$ of τ_0 such that

$$M(\bar{\tau}) = 0 \tag{32}$$

and

$$\left.\frac{\mathrm{d}M(\tau_0)}{\mathrm{d}\tau_0}\right|_{\bar{\tau}} \neq 0\,. \tag{33}$$

Before applying the Melnikov theorem, it is first necessary to calculate the Melnikov function for both branches of the homoclinic solution. By equations (24) and (31) and noting that $v_-^0 \sin u_-^0 = v_+^0 \sin u_+^0$

$$\begin{aligned} M_\pm(\tau_0) &= \int_{-\infty}^{\infty} (v_+^0(\tau))^2 - \sigma v_+^0(\tau) \sin u_+^0(\tau) \cos^2(\lambda_5 r^2(\tau+\tau_0))\,\mathrm{d}\tau \\ &= \int_{-\pi/2}^{\pi/2} v_+^0(u)\,\mathrm{d}u - \sigma \int_{-\infty}^{\infty} v_+^0(\tau) \sin u_+^0(\tau) \cos^2(\lambda_5 r^2(\tau+\tau_0))\,\mathrm{d}\tau \\ &= I_1 - \sigma I_2\,. \end{aligned} \tag{34}$$

By (29), with $H = 0$,

$$\begin{aligned} I_1 &= \sqrt{A} \int_{-\pi/2}^{\pi/2} \cos u\,\mathrm{d}u \\ &= 2\sqrt{A}\,. \end{aligned} \tag{35}$$

Expanding the squared cosine term in I_2 and noting that by (30), $v_+^0(\tau)\sin u_+^0(\tau)$ is odd in τ, gives

$$\begin{aligned} I_2 &= -\sqrt{A}\sin(2\lambda_5 r^2\tau_0) \int_0^{\infty} \operatorname{sech}(\sqrt{A}\tau)\tanh(\sqrt{A}\tau)\sin(2\lambda_5 r^2\tau)\,\mathrm{d}\tau \\ &= -\sin(2\lambda_5 r^2\tau_0)\Big\{\Big[-\operatorname{sech}(\sqrt{A}\tau)\sin(2\lambda_5 r^2\tau)\Big]_0^{\infty} \\ &\quad + 2\lambda_5 r^2 \int_0^{\infty} \cos(2\lambda_5 r^2\tau)\operatorname{sech}(\sqrt{A}\tau)\,\mathrm{d}\tau\Big\} \\ &= -\frac{\pi\lambda_5 r^2}{\sqrt{A}}\sin(2\lambda_5 r^2\tau_0)\operatorname{sech}\left(\frac{\lambda_5 r^2\pi}{\sqrt{A}}\right) \end{aligned} \tag{36}$$

the final integrand being evaluated via an expression given in [12, p 503]. Therefore

$$\sqrt{A}M_\pm(\tau_0) = 2A + \sigma\pi\lambda_5 r^2 \sin(2\lambda_5 r^2\tau_0)\operatorname{sech}\left(\frac{\lambda_5 r^2\pi}{\sqrt{A}}\right). \tag{37}$$

The form of the above Melnikov function makes application of the Melnikov theorem particularly simple. Obviously simple zeros and therefore chaos in the sense of Smale horseshoes exist whenever

$$2A < |\sigma|\pi\lambda_5 r^2 \operatorname{sech}\left(\frac{\lambda_5 r^2\pi}{\sqrt{A}}\right). \tag{38}$$

4. Discussion

The above Melnikov analysis demonstrates that chaos exists within the sample whenever the following holds:

$$h_1(R) < h_2(R) \tag{39}$$

where

$$R = \frac{\lambda_5 r^2 \pi}{\sqrt{A}} \tag{40}$$

$$h_1(R) = \cosh R \tag{41}$$

$$h_2(R) = c\, R \tag{42}$$

$$c = \frac{|\sigma|}{2\sqrt{A}}\,. \tag{43}$$

In these equations the variables of interest physically are R and c. R is a measure of distance from the apex of the wedge and by reference to (23) the gradient, c, of $h_2(R)$ varies with the applied voltage; c increases from zero when the peak applied voltage increases from zero. From figure 2 it is apparent that for a low applied voltage (and therefore low gradient c) $h_2(R)$ will not intersect $h_1(R)$ with the consequence that chaotic behaviour is not present. At a critical value of the peak applied voltage (V_c) $h_2(R)$ becomes tangential to $h_1(R)$ at a point $\bar{R}$. For a peak applied voltage above this critical value the gradient of $h_2(R)$ increases similarly ensuring that $h_1(R)$ and $h_2(R)$ intersect at two points

$$R_0 = \frac{\lambda_5 r_0^2 \pi}{\sqrt{A}} \tag{44}$$

and

$$R_1 = \frac{\lambda_5 r_1^2 \pi}{\sqrt{A}}\,. \tag{45}$$

Between these two points our Melnikov analysis dictates that chaos exists.

To determine the critical peak applied voltage we first find the value c_t of the gradient of $h_2(R)$ such that $h_2(R)$ is tangential to $h_1(R)$, this in turn will lead directly to the critical

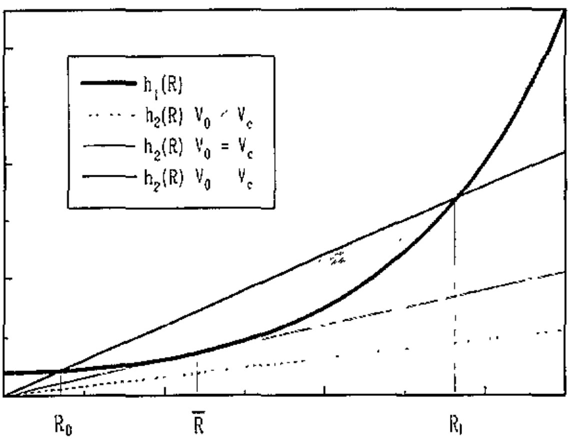

Figure 2. Qualitative plots of $h_1(R)$ and $h_2(R)$ for peak voltages V_0 below, equal to, and above the critical voltage V_c. For $V_0 > V_c$ chaotic instabilities occur for values of R between R_0 and R_1.

peak applied voltage V_c. In order to do this we note that by simple differentiation the value of R which gives the minimum distance between $h_1(R)$ and $h_2(R)$ is

$$R' = \sinh^{-1} c\,. \tag{46}$$

Therefore c_t is the value of c such that $j(c) \triangleq h_1(R') - h_2(R')$ is zero. This gives the following expression for c_t:

$$\begin{aligned} j(c_t) &= \cosh(\sinh^{-1} c_t) - c_t \sinh^{-1} c_t \\ &= \sqrt{c_t^2 + 1} - c_t \sinh^{-1} c_t = 0\,. \end{aligned} \tag{47}$$

We solve this by use of an iteration of the form

$$\begin{aligned} c_{n+1} &= c_n - \frac{j(c_n)}{j'(c_n)} \\ &= \frac{\sqrt{c_n^2 + 1}}{\sinh^{-1} c_n} \end{aligned} \tag{48}$$

to find that

$$c_t = 1.508\,80\,. \tag{49}$$

Hence for values of the peak applied voltage V_0 that force c above 1.508 80 we have chaotic behaviour. By (5), (23), (43) and (49) we gain the critical voltage above which chaos occurs as

$$V_c = \frac{4.913\,\eta\sqrt{\omega}\,A^{1/4}}{\sqrt{|\epsilon_a|\epsilon_0}\,\sin\theta}\,. \tag{50}$$

As can be visualized from figure 2 the resultant chaos exists whenever R lies between R_0 and R_1 which are related to the distance from the apex of the wedge by (44) and (45). The chaotic region lies within a band in the sample which grows as the peak voltage is increased above V_c. Such chaotic regions may represent areas of unpredictable and non-repeating twist wall motion in the sample. Hence we envisage that by experimental observation of the boundaries of the chaotic region with respect to r along with knowledge of the value of A (possibly gained from (50)), it may be feasible by numerical investigation of (39) to gain a value for the viscosity coefficient λ_5. However, due to the simplification made to derive equation (30), which in turn made the calculations more tractable, we have no method of predicting the boundaries of the chaotic region with respect to α. Indeed it is entirely possible that the chaotic region tapers towards either limit in r, especially the lower limit due to the narrowness of the wedge. In this case experimental values of r_0 and r_1 may be difficult to obtain to a reasonable accuracy. Thus it is hoped that by careful experimental observation values for V_c and λ_5 could be determined.

Finally we shall mention the assumed form of the electric field. The oscillatory part is quite straightforward; however, the peak applied voltage is linked to the angular frequency. This of course does not pose any particular problems although it may seem unusual and places a restriction on the strength of the applied field: a large peak applied voltage could affect the resulting analysis. For a high-voltage experiment we note that any resultant layer distortions would negate this analysis as we have assumed that the layers remain intact; this point aside, the form of a strong voltage would be better represented by

$$V(t) = V\cos(\omega t) \tag{51}$$

than by (6). At first glance one may expect the analysis resulting from equation (51) to follow the lines of those by Wiggins [13] or Hastings and McLeod [14]. These articles are concerned with the motion of a forced pendulum but unlike previous Melnikov analyses of this problem, the amplitude of the forcing term is not necessarily small. This appears to be similar to replacing equation (6) with (51) in our problem. However, with this modification the manipulations we employ between equations (17) and (24) give a system which cannot be brought into a suitable form to apply the extension to Melnikov's theory as proposed by Wiggins [13]. Hence our present analysis is inappropriate for strong fields of the above form although the evidence suggests that chaos would indeed exist.

Appendix

The value of $\mathbf{\Pi}^c$ is defined by Leslie *et al* [3] as

$$\mathbf{\Pi}_i^c = \nabla \cdot \left(\frac{\partial w}{\partial \nabla c}\right) - \frac{\partial w}{\partial c} \tag{A.1}$$

where w is the bulk energy of the system. In terms of $\boldsymbol{a}$ and $\boldsymbol{c}$ and their gradients we adopt a simplified form of the energy proposed by Leslie *et al* [15] and assume a uniaxial dielectric response [1] to give

$$\begin{aligned}
2w = {} & A_{21}(\nabla\cdot\boldsymbol{a})^2 + B\{(\boldsymbol{a}\cdot\nabla\times\boldsymbol{c})^2 + (\nabla\cdot\boldsymbol{c})^2\} \\
& + B_3\{(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})^2 + (\boldsymbol{c}\cdot\nabla\times\boldsymbol{c})^2 - (\nabla\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})\} \\
& + 2A\{(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})^2 - (\nabla\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})\} \\
& - 2B_{13}\{(\boldsymbol{a}\cdot\nabla\times\boldsymbol{c})(\boldsymbol{c}\cdot\nabla\times\boldsymbol{c}) + (\nabla\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})\} \\
& + 2(C_1+C_2)(\nabla\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c}) - 2C_2(\nabla\cdot\boldsymbol{a})(\nabla\cdot\boldsymbol{c}) \\
& - \epsilon_a\epsilon_0(\boldsymbol{a}\cdot\boldsymbol{E}\cos\theta + \boldsymbol{c}\cdot\boldsymbol{E}\sin\theta)^2. \\
= {} & \frac{1}{r^2}\left(A_{21} + B\left(\frac{\mathrm{d}\phi}{\mathrm{d}\alpha}\right)^2 - 2A\sin^2\phi\,\cos^2\phi + 2(C_1+C_2)\cos\phi\,\sin^2\phi\frac{\mathrm{d}\phi}{\mathrm{d}\alpha}\right. \\
& \left. - 2C_2\cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} - \epsilon_0\epsilon_a E^2\sin^2\theta\,\sin^2\phi\right).
\end{aligned} \tag{A.2}$$

Here θ is the constant smectic C tilt angle mentioned in the introduction. A term in the electrical energy which is independent of orientation (and hence does not occur later) has been ignored. The constants ϵ_0 and ϵ_a represent the permittivity of free space and the dielectric anisotropy of the liquid crystal (the difference in relative permittivities parallel and perpendicular to the n-director) respectively. The anisotropy ϵ_a can be positive or negative, depending upon the specific liquid crystal in question. The A, B and C parameters are elastic constants where in the notation of Leslie *et al* [15] we have set

$$B_1 = B_2 = B \qquad A_{11} + A_{12} = A_{11} + A_{21} = A\,. \tag{A.3}$$

The elastic constants appearing here are the same as those introduced by the Orsay Group [16] except that $A_{11} = -\frac{1}{2}A_{11}^{\mathrm{Orsay}}$ and $C_1 = -C_1^{\mathrm{Orsay}}$. A brief physical interpretation of the deformations related to each of the elastic constants has been given by Carlsson *et al* [4], from whose work it can also be seen that

$$B > 0 \qquad \text{and} \qquad A > 0\,. \tag{A.4}$$

The general form of $\tilde{g}^c$ is given in [3]; however, our assumed absence of bulk flow reduces this contribution to

$$\tilde{g}^c = -2\left(\lambda_5\frac{\partial c}{\partial t} + \tau_5\frac{\partial a}{\partial t}\right) = -2\lambda_5\left(0, \cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}t}, -\sin\phi\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) \qquad \text{(A.5)}$$

where λ_5 is a positive viscosity coefficient [3].

For the evaluation of $\mathbf{\Pi}^c$ the non-negative components of the energy give rise to the following expression which can be verified (for further details see [17]):

$$\begin{aligned}\mathbf{\Pi}^c = {} & B\nabla\{(\nabla\cdot\boldsymbol{c}) - \nabla\times((\boldsymbol{a}\cdot\nabla\times\boldsymbol{c})\boldsymbol{a})\} \\ & +2A\{(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})(\boldsymbol{a}\times\nabla\times\boldsymbol{c}) - \nabla\times((\boldsymbol{b}\cdot\nabla\times\boldsymbol{c})\boldsymbol{b})\} + \nabla\times((\nabla\cdot\boldsymbol{a})\boldsymbol{b}) \\ & -(\nabla\cdot\boldsymbol{a})(\boldsymbol{a}\times\nabla\times\boldsymbol{c})\} \\ & +(C_1+C_2)\{\nabla(\boldsymbol{b}\cdot\nabla\times\boldsymbol{c}) - \nabla\times((\nabla\cdot\boldsymbol{c})\boldsymbol{b}) + (\nabla\cdot\boldsymbol{c})(\boldsymbol{a}\times\nabla\times\boldsymbol{c})\} \\ & -C_2\nabla(\nabla\cdot\boldsymbol{a}) + \epsilon_a\epsilon_0(\boldsymbol{a}.\boldsymbol{E}\cos\theta + \boldsymbol{c}\cdot\boldsymbol{E}\sin\theta)\boldsymbol{E}\sin\theta\,. \end{aligned} \qquad \text{(A.6)}$$

From this we may calculate the required second and third components of $\mathbf{\Pi}^c$ via the ansatz (3) as

$$\Pi_2^c = \frac{1}{r^2}\left\{B\left(-\sin\phi\left(\frac{\mathrm{d}\phi}{\mathrm{d}\alpha}\right)^2 + \cos\phi\frac{\mathrm{d}^2\phi}{\mathrm{d}\alpha^2}\right) + 4A\sin\phi\cos^2\phi\right\} \qquad \text{(A.7)}$$

$$\Pi_3^c = \frac{B}{r^2}\left(-\cos\phi\left(\frac{\mathrm{d}\phi}{\mathrm{d}\alpha}\right)^2 - \sin\phi\frac{\mathrm{d}^2\phi}{\mathrm{d}\alpha^2}\right). \qquad \text{(A.8)}$$

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